# The Utility Frontier

Any allocation $(\mathbf{x}^i)_1^n$ to a set $N = \{1, \ldots, n\}$ of individuals with utility functions $u^1(\cdot), \ldots, u^n(\cdot)$ yields a profile $(u_1, \ldots, u_n)$ of resulting utility levels, as depicted in Figure 1 for the case $n = 2$. (Throughout this set of notes, in order to distinguish between utility *functions* and utility *levels*, I'll use superscripts for the functions and subscripts for the resulting levels, as I've done in the preceding sentence and in Figure 1.) Let's formally define the function that accomplishes this:

$$U : \mathbb{R}_+^{n\ell} \to \mathbb{R}^n \text{ is defined by } U\left((\mathbf{x}^i)_N\right) = \left(u^1(\mathbf{x}^1), \ldots, u^n(\mathbf{x}^n)\right) \tag{$*$}$$

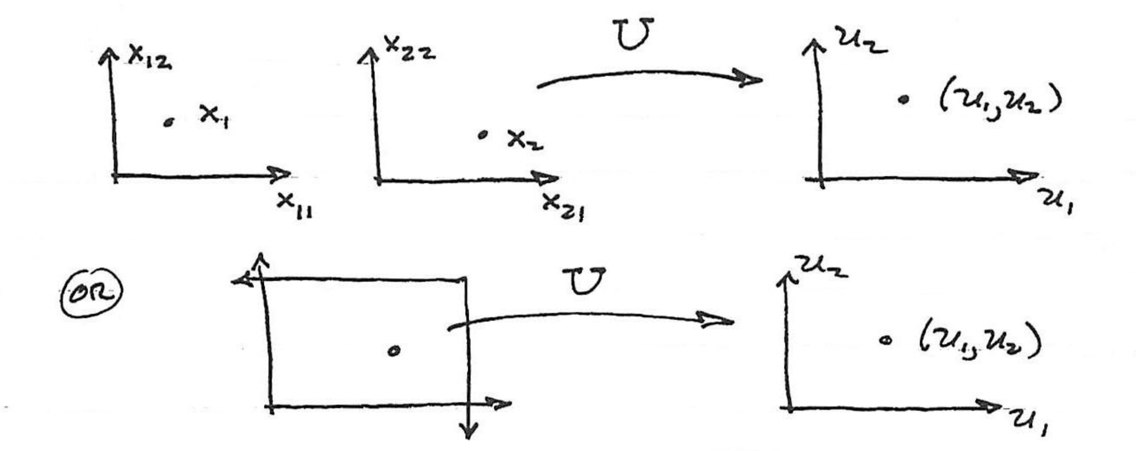



Figure 1

Let $\mathcal{F}$ denote the set of feasible allocations — *i.e.*, those that satisfy $\sum_1^n \mathbf{x}^i \leqq \mathring{\mathbf{x}}$. The set of **feasible utility profiles** is the image under $U$ of the set of all feasible allocations, *i.e.*, $U(\mathcal{F})$:

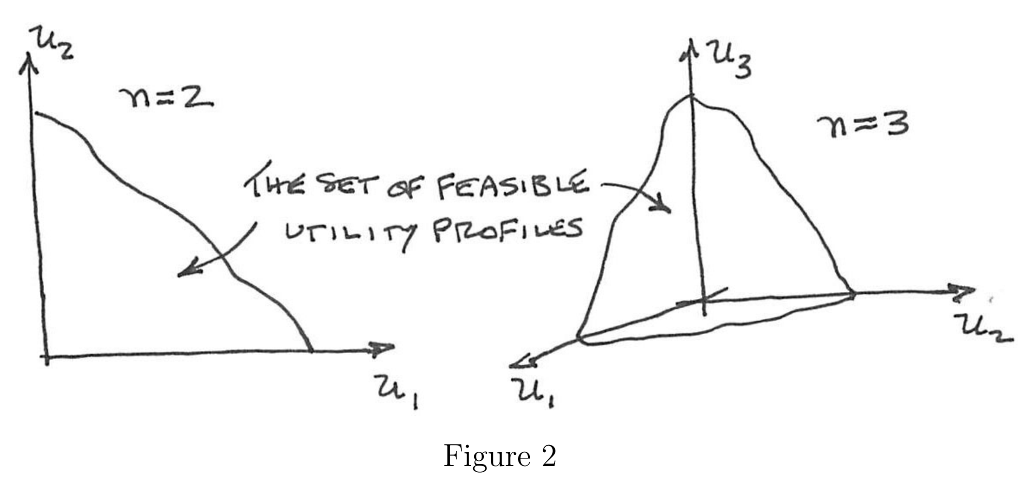



Figure 2

The Pareto efficient allocations are clearly the ones that get mapped by $U$ to the "northeast" part of the boundary of the set of feasible utility profiles. (More accurately, to those points $\mathbf{u}$ on the boundary of $U(\mathcal{F})$ for which there are no other points in $U(\mathcal{F})$ lying to the northeast). This northeast part of the set $U(\mathcal{F})$ is called the **utility frontier**, which we'll denote by UF. It consists of the utility profiles $\mathbf{u} = (u_1, \ldots, u_n)$ that are maximal in $U(\mathcal{F})$ with respect to the preorder $\geqq$ on $\mathbb{R}^n$:

$$\mathbf{u} = (u_1, \ldots, u_n) \in \text{UF if and only if}$$
$$\mathbf{u} \in U(\mathcal{F}) \text{ and there is no } \mathbf{u}' \in U(\mathcal{F}) \text{ that satisfies } \forall i : u_i' \geqq u_i \;\&\; \exists i : u_i' > u_i \ .$$

Equivalently, UF is the image under $U$ of the set of Pareto allocations:

$$\text{UF} = U(\mathcal{P}), \text{ where } \mathcal{P} \text{ is the set of Pareto allocations in } \mathbb{R}_+^{n\ell}.$$

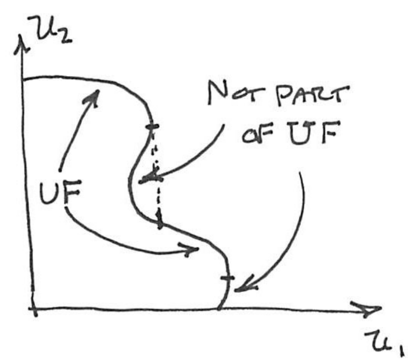



Figure 3

Note that the alternatives over which the individuals have utility functions needn't be allocations: we could replace the set $\mathbb{R}_+^{n\ell}$ of allocations with an arbitrary set $X$ of alternatives $x$, and $(*)$ would become

$$U : X \to \mathbb{R}^n \text{ is defined by } U(x) = \big(u^1(x), \ldots, u^n(x)\big)$$

Figure 2 would still look the same: it would be $U(X)$, or $U(\mathcal{F})$, the image under $U$ of either $X$ or $\mathcal{F}$; and Figure 3 would be the same, the image under $U$ of the set of Pareto efficient alternatives.

The utility frontier is a surface in $\mathbb{R}^n$, and it could be expressed as the set of profiles $(u_1, \ldots, u_n)$ that satisfy the equation $h(u_1, \ldots, u_n) = 0$ for some function $h$, or

$$u_1 = g(u_2, \ldots, u_n) \qquad (**)$$

for some function $g$. In the equation $(**)$, the function $g$ tells us, for given utility levels $u_2, \ldots, u_n$ for $n-1$ individuals, what is the maximum utility level $u_1$ that's feasible for the remaining individual. In other words, $g$ is the *value function* for the problem (P-Max), in which the utility levels $u_2, \ldots, u_n$ are parameters and we solve for the allocation $(\mathbf{x}^i)_1^n$ in which $\mathbf{x}^1$ maximizes $u^1(\cdot)$ subject to all other individuals $i = 2, \ldots, n$ receiving at least the utility level $u_i$ (recall that we're using $u^i$ to denote utility *functions* and $u_i$ to denote utility *levels*!):

$$\begin{array}{rcll} \max\limits_{(x_k^i)\in\mathbb{R}_+^{nl}} u^1(\mathbf{x}^1) & & & \\ \text{subject to} \quad x_k^i & \geqq & 0, \quad i=1,...,n, \quad k=1,...,l & \\ \sum\limits_{i=1}^n x_k^i & \leqq & \mathring{x}_k, \quad k=1,...,l & \\ u^i(\mathbf{x}^i) & \geqq & u_i, \quad i=2,...,n. & \end{array} \tag{P-Max}$$

## The Solution Function and the Value Function for a Maximization Problem

Consider the maximization problem

$$\max_x f(x;\alpha) \;\text{ subject to }\; G(x;\alpha) \leqq \mathbf{0}. \tag{$\mathbf{P}$}$$

Note that we're maximizing over $x$ and not over $\alpha$ — $x$ is a variable in the problem (typically a vector or $n$-tuple of variables) and $\alpha$ is a parameter (typically a vector or $m$-tuple of parameters). The parameters may appear in the objective function and/or the constraints, if there are any constraints. We associate the following two functions with the maximization problem (**P**):

the **solution function**: $\quad x = x(\alpha), \quad$ and

the **value function**: $\quad v(\alpha) := f(x(\alpha))$ .

The solution function gives the solution $x$ as a function of the parameters; the value function gives the value of the objective function as a function of the parameters.

**Example 1:** The consumer maximization problem (CMP) in demand theory,

$$\max_{\mathbf{x}\in\mathbb{R}_+^\ell} u(\mathbf{x}) \;\text{ subject to }\; \mathbf{p}\cdot\mathbf{x} \leqq w.$$

Here $\alpha$ is the $(\ell+1)$-tuple $(\mathbf{p}; w)$ consisting of the price-list $\mathbf{p}$ and the consumer's wealth $w$.

The solution function is the consumer's demand function $\mathbf{x}(\mathbf{p}; w)$ .

The value function is the consumer's indirect utility function $v(\mathbf{p}; w) = u(\mathbf{x}(\mathbf{p}; w))$ .

**Example 2:** The firm's cost-minimization (*i.e.*, expenditure-minimization) problem,

$$\min_{\mathbf{x}\in\mathbb{R}^\ell_+} E(\mathbf{x};\mathbf{w}) = \mathbf{w}\cdot\mathbf{x} \ \text{ subject to } \ F(\mathbf{x}) \geqq y.$$

Here $F$ is the firm's production function; $\mathbf{x}$ is the $\ell$-tuple of input levels that will be employed; $E(\mathbf{x};\mathbf{w})$ is the resulting expenditure the firm will incur; and $\alpha$ is the $(\ell+1)$-tuple $(y;\mathbf{w})$ consisting of the proposed level of output, $y$, and the $\ell$-tuple $\mathbf{w}$ of input prices.

The solution function is the firm's input demand function $\mathbf{x}(y;\mathbf{w})$ .

The value function is the firm's cost function $C(y;\mathbf{w}) = E(\mathbf{x}(y;\mathbf{w});\mathbf{w})$ .

**Example 3:** The Pareto problem (P-Max),

$$\max_{\mathbf{x}\in\mathcal{F}} u^1(\mathbf{x}^1) \ \text{ subject to } \ u^2(\mathbf{x}^2) \geqq u_2\,, \ \ldots\ , \ u^n(\mathbf{x}^n) \geqq u_n,$$

where $\mathcal{F}$ is the feasible set $\{\mathbf{x}\in\mathbb{R}^{n\ell}_+ \mid \sum_1^n \mathbf{x}^i \leqq \mathring{\mathbf{x}}\}$ . Here $\alpha$ is the $(n-1)$-tuple of utility levels $u_2,\ldots,u_n$.

The solution function is $\mathbf{x}(u_2,\ldots,u_n)$, which gives the Pareto allocation as a function of the utility levels $u_2,\ldots,u_n$ .

The value function is $u^1(\mathbf{x}(u_2,\ldots,u_n))$ , which gives the maximum attainable utility level $u_1$ as a function of the utility levels $u_2,\ldots,u_n$ .

The value function therefore describes the utility frontier for the economy $((u^i)_1^n,\mathring{\mathbf{x}})$, as depicted in Figure 2.

EXAMPLE:

$N = \{1, \ldots, n\}$; $u_i(x_i, y_i) = x_i y_i$, $\forall i \in N$;

TOTAL ENDOWMENT IS $(\mathring{x}, \mathring{y})$.

PARETO EFFICIENCY REQUIRES THAT, FOR SOME NUMBER $r$:

$$\frac{y_1}{x_1} = \frac{y_2}{x_2} = \cdots = \frac{y_n}{x_n} = r; \quad \text{i.e., } y_i = r x_i, \ \forall i \in N.$$

$$\therefore \mathring{y} = r\mathring{x} \quad [\mathring{y} = \Sigma y_i = \Sigma r x_i = r \Sigma x_i = r\mathring{x}],$$

$$\text{i.e., } \boxed{r = \frac{\mathring{y}}{\mathring{x}}}$$

AT ANY EFFICIENT ALLOCATION, THEN, WE MUST HAVE, $\forall i \in N$:

$$u_i(x_i, y_i) = x_i y_i = (x_i)(r x_i) = r x_i^2;$$

$$\text{i.e., } \sqrt{u_i} = \sqrt{r}\, x_i.$$

$$\therefore \sum_{i \in N} \sqrt{u_i} = \sqrt{r} \sum_{i \in N} x_i = \sqrt{r}\, \mathring{x}.$$

$$\therefore \sum_{i \in N} \sqrt{u_i} = \sqrt{r}\, \mathring{x} = \frac{\sqrt{\mathring{y}}}{\sqrt{\mathring{x}}} \mathring{x} = \sqrt{\mathring{x}\mathring{y}}.$$

IN OTHER WORDS, THE UTILITY FRONTIER IS THE EQUATION

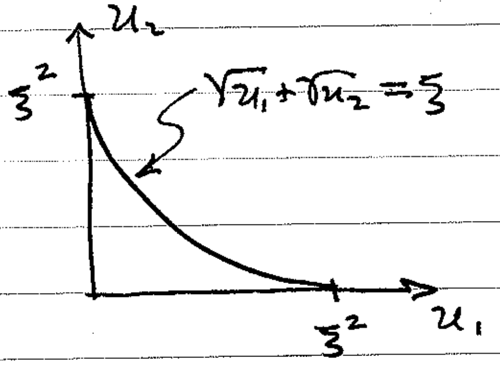
$$\sum_{i \in N} \sqrt{u_i} = \sqrt{\mathring{x}\mathring{y}},$$

OR ITS GRAPH IN $\mathbb{R}^n$.

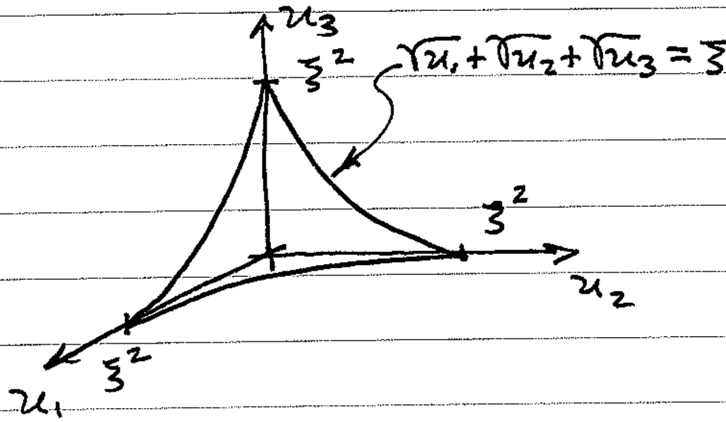
DANGER: THE UTILITY FRONTIER HAS THIS FORM IN THIS EXAMPLE, WHERE ALL UTILITY FUNCTIONS ARE OF THE FORM $u(x,y) = xy$.

Let $\xi := \sqrt{\mathring{x}\mathring{y}}$.

$n=2$:




$n=3$:



EXAMPLE: (THE UTILITY FRONTIER AND THE CORE)

$N = \{1,2,3\}$ ; $u_i(x_{i1}, x_{i2}) = x_{i1} \, x_{i2}$ , $i = 1,2,3.$

$\overset{\circ}{x}_1 = \overset{\circ}{x}_2 = (30,0)$ ; $\overset{\circ}{x}_3 = (0,60).$

PROPOSAL: $\hat{x}_i = (20,20)$, $i = 1,2,3.$ $u_i(\hat{x}_i) = 400$, $i = 1,2,3.$

CLEARLY, $(\hat{x}_i)_N$ IS PARETO EFFICIENT AND INDIVIDUALLY ACCEPTABLE.

BUT $\{1,3\}$ CAN IMPROVE UPON $(\hat{x}_i)_N$ VIA $(\tilde{x}_i)_{\{1,3\}}$,

WHERE $\tilde{x}_1 = \tilde{x}_3 = (15,30)$ :

WE HAVE $\tilde{x}_1 + \tilde{x}_3 = (30,60) = \overset{\circ}{x}_1 + \overset{\circ}{x}_3$ AND $u_1(\tilde{x}_1) = u_2(\tilde{x}_2) = 450.$

THE COALITION $\{2,3\}$ COULD IMPROVE IN THE SAME WAY.

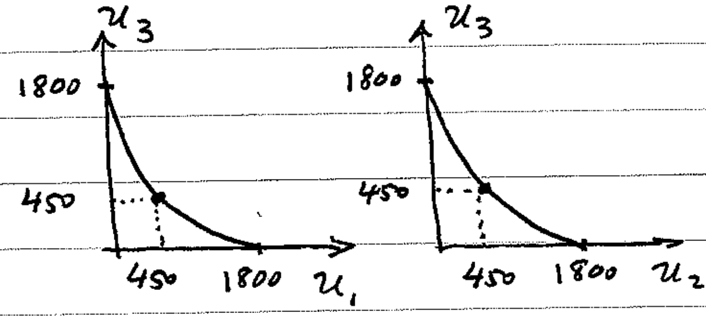
IN FACT, IT IS CLEAR THAT UNLESS A PROPOSAL $(x_i)_N$ GIVES BOTH $u_1 \geqq 450$ AND $u_2 \geqq 450$, OR ELSE $u_3 \geqq 450$, THEN EITHER $\{1,3\}$ OR $\{2,3\}$ WILL BE ABLE TO UNILATERALLY IMPROVE UPON $(x_i)_N$: ANY PROPOSAL THAT $u_3 < 450$ AND EITHER $u_1 < 450$ OR $u_2 < 450$ CAN BE IMPROVED UPON BY $\{1,3\}$ OR $\{2,3\}$ AS ABOVE.

IN FACT, THE UTILITY FRONTIERS FOR $\{1,3\}$ AND $\{2,3\}$ ARE $\sqrt{u_1} + \sqrt{u_3} = \sqrt{(30)(60)} = \sqrt{1800} = 30\sqrt{2}$

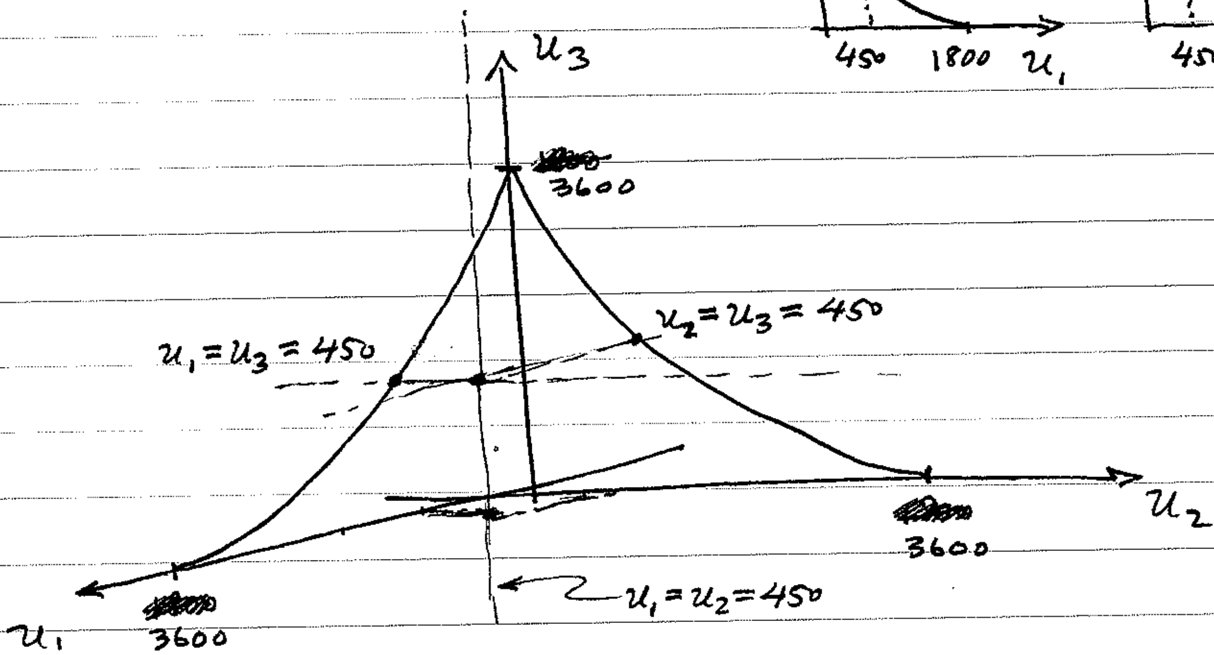
AND $\sqrt{u_2} + \sqrt{u_3} = \sqrt{(30)(60)} = \sqrt{1800} = 30\sqrt{2}.$

SINCE PARETO EFFICIENCY IN THIS EXAMPLE REQUIRES $x_{i1} = x_{i2} = z_i$, SAY, FOR $i = 1,2,3$, WE HAVE

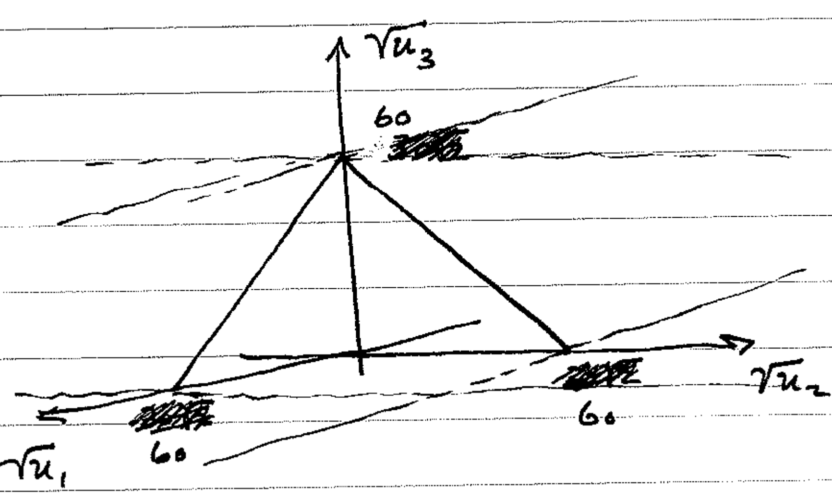
$$z_1 + z_3 \geqq 30\sqrt{2} \approx 42.4 \text{ AND } z_2 + z_3 \geqq 30\sqrt{2} \approx 42.4.$$



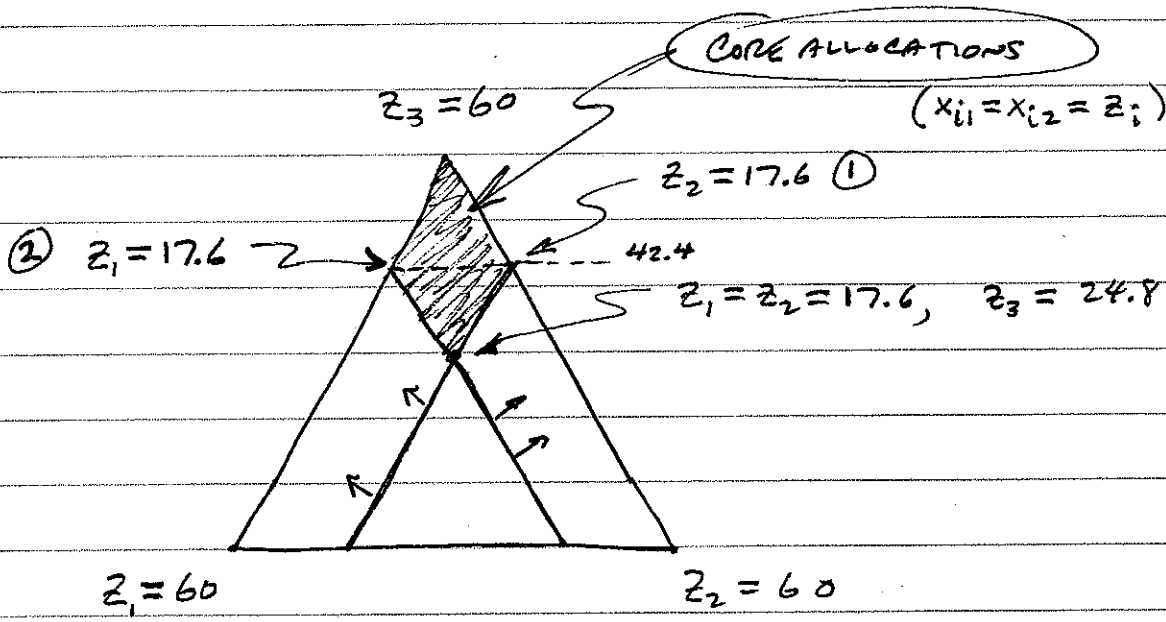
$u_3$
1800
450
450
1800
$u_1$
$u_3$
1800
450
450
1800
$u_2$

$u_3$
3600
$u_2 = u_3 = 450$
$u_1 = u_3 = 450$
$u_2$
3600
$u_1 = u_2 = 450$
$u_1$
3600

$\sqrt{u_3}$
60
$\sqrt{u_2}$
60
$\sqrt{u_1}$
60




① $z_1 + z_3 \geqq 42.4$ i.e., $z_2 \leqq 17.6$

② $z_2 + z_3 \geqq 42.4$ i.e., $z_1 \leqq 17.6$

$\therefore$ $z = (20, 20, 20)$ IS <u>NOT</u> IN THE CORE